COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA 11 (1) (1991) 55–61

SPANNING TREES WITH BOUNDED DEGREES

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Received July 14, 1988
Revised May 4, 1990

Let s and k be positive integers. We prove that if G is a k-connected graph containing no independent set with $ks+2$ vertices then G has a spanning tree with maximum degree at most $s+1$. Moreover if $s \geq 3$ and the independence number $\alpha(G)$ is such that $\alpha(G) \leq 1+k(s-1)+c$ for some $0 \leq c \leq k$ then G has a spanning tree with no more than c vertices of degree $s+1$.

A basic result in graph theory asserts that any connected graph has a spanning tree. Some research has been done to obtain sufficient conditions for a graph to contain spanning trees of a special kind. See, for instance, [1, 2, 3 and 4].

Our starting point is a well known theorem due to V. Chvátal and P. Erdős [1] which asserts that any k-connected graph with independence number $\alpha \leq k+1$ has a hamiltonian path. In [2], S. Win gives a proof of a conjecture of M. Las Vergnas which generalizes this theorem; his result states that every k-connected graph with independence number $\alpha \leq k+c$ contains a spanning tree with no more than $c+1$ terminal vertices.

In this article we give another generalization of the same theorem, namely, if G is a k-connected graph with independence number $\alpha \leq 1+ks$, for some $s \geq 1$ then G has a spanning tree T with no vertices of degree larger than $s+1$ (theorem 3); moreover we are able to bound the number of vertices with degree $s+1$ in T (theorem 2).

We start by setting some notation and establishing a lemma which will be useful in the proof of the main results.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any subset U of $V(G)$ we denote by $G-U$ the graph obtained from G by deleting all the vertices in U. Analogously, for a subset L of $E(G)$, $G-L$ will denote the graph obtained from G by deleting the edges in L. If e is not an edge of G then $G+e$ is the graph obtained by adding the edge e to G.

An outdirected tree $\vec{T}$ is a rooted tree in which all the edges are directed away from the root. Whenever $\vec{T}$ is an outdirected tree with vertex set $V(\vec{T})$ and arc set $A(\vec{T})$, given a subset U of $V(T)$, we shall denote by $N^+(U)$ the set of vertices $w \in V(\vec{T})$ for which there is an arc $uw \in A(\vec{T})$ for some $u \in U$. We define $N^-(U)$

AMS subject classification (1980): 05 C 05
Keywords: Spanning tree, maximum degree, k-connected graph and independence number.

in an analogous way. For any $u \in V(T)$, the set $N^-(\{u\})$ consists of a unique vertex which will be denoted by u^-.

Given an outdirected tree $\vec{T}$, we denote by T the corresponding undirected tree; by $V_1(T)$ the set of terminal vertices of T and if u and v are any vertices of T then T_{uv} is the unique path in T joining u and v.

Lemma 1. *Let $\vec{T}$ be an outdirected tree, and let R and B be disjoint subsets of $V(T)$ such that $(R \cup B) \cap V_1(T) = \emptyset$ and $N^+(R \cup B) \cap R = \emptyset$. For each $b \in B$ let x_b be any fixed vertex in $N^+(b)$ and let $X(B) = \{x_b : b \in B\}$. There exists a one to one function $\phi : N^+(R) \cup (N^+(B) \setminus X(B)) \to V_1(T) \cup N^-(R)$ which satisfies:*

1) *If $u \in N^+(R) \cup (N^+(B) \setminus X(B))$ then $T_{u\phi(u)}$ is u to $\phi(u)$ directed in $\vec{T}$.*
2) *$T_{u\phi(u)}$ and $T_{v\phi(v)}$ are vertex disjoint whenever $u \neq v$.*
3) *If $u \neq v$ then at least one of u^- and v^- is a vertex of $T_{\phi(u)\phi(v)}$.*
4) *The range of ϕ is an independent set in T.*

Proof. Let $Y = \{ba \in A(\vec{T}) : b \in B \text{ and } a \neq x_b\}$. Clearly each weak component of the directed graph $\vec{H} = (\vec{T} - R) - Y$ is an outdirected tree; moreover each $u \in N^+(R) \cup (N^+(B) \setminus X(B))$ is the root of the weak component $\vec{H}_u$ of $\vec{H}$ that contains u and each terminal vertex of $\vec{H}_u$ is either a terminal vertex of $\vec{T}$ or is in $N^-(R)$, therefore we can define ϕ by letting $\phi(u)$ be any terminal vertex of $\vec{H}_u$; conditions 1 and 2 are satisfied by construction.

Let r be the root of $\vec{T}$, and let u and v be two different vertices in $N^+(R) \cup (N^+(B) \setminus X(B))$. The paths $T_{u\phi(u)}$ and $T_{v\phi(v)}$ are contained in $T_{r\phi(u)}$ and $T_{r\phi(v)}$, respectively. Let $w \in V(T_{r\phi(u)}) \cap V(T_{r\phi(v)})$ be such that $T_{\phi(u)\phi(v)} = T_{w\phi(u)} \cup T_{w\phi(v)}$. By condition 2, $T_{u\phi(u)}$ and $T_{v\phi(v)}$ are vertex disjoint; hence if $w = \phi(u)$ then $v^- \in V(T_{\phi(u)\phi(v)})$; if $w = \phi(v)$ then $u^- \in V(T_{\phi(u)\phi(v)})$ and if $\phi(u) \neq w \neq \phi(v)$ then $u^- \in V(T_{w\phi(u)})$ and $v^- \in V(T_{w\phi(v)})$.

Finally let us suppose that the arc $\phi(u)\phi(v)$ is in $A(\vec{T})$. Since $T_{u\phi(u)}$ and $T_{v\phi(v)}$ are vertex disjoint, then $v = \phi(v)$ and therefore $v^- = \phi(u)$. By construction v^- is in $R \cup B$ and $\phi(u)$ is in $V_1(T) \cup N^-(R)$ but $(R \cup B) \cap (V_1(T) \cup N^-(R)) = \emptyset$. ∎

We can now proceed to prove our main result.

Theorem 2. *Let G be a k-connected graph with independence number α, and let s and c be integers with $3 \leq s$ and $0 \leq c \leq k$. If $\alpha \leq 1 + k(s-1) + c$ then G has a spanning tree T with degrees bounded above by $s+1$ and with at most c vertices of degree $s+1$.*

Proof. We call a subtree of G a $(s+1, c)$-subtree if the maximum degree and the number of vertices with degree $s+1$ in the tree do not exceed $s+1$ and c, respectively.

Let T be a $(s+1, c)$-subtree of G with the maximum possible number of vertices. If T is not a spanning tree we choose w to be any vertex of G not in T. Let $P = \{\pi_1, \pi_2, \ldots, \pi_\ell\}$ be a maximum collection of w to T paths in G, pairwise disjoint apart from vertex w. For each i let r_i be the unique vertex of π_i in T.

Assume $d_T(r_1) \leq d_T(r_2) \leq \ldots \leq d_T(r_\ell)$. By the choice of T we have $s \leq d_T(r_1)$ and $d_T(r_\ell) \leq s+1$; and by a well known variation of Menger's theorem we know that ℓ is at least k.

Call $R = \{r_1,\ r_2,\ \ldots,\ r_\ell\}$ and let t be the number of vertices in R with degree s in T. Let $B = \{b_1,\ b_2,\ \ldots,\ b_m\}$ be the set of vertices in $V(T) \setminus R$ having degree $s+1$ in T. If $t > 0$, in particular $d_T(r_1) = s$, and then $T \cup \pi_1$ is a subtree of G having degrees at most $s+1$. Since T is a maximum $(s+1,\ c)$-subtree of G, then it must contain exactly c vertices of degree $s+1$; therefore $m = c - \ell + t$. When $t = 0$ then $d_T(r_i) = s+1$ for $i = 1,\ 2,\ \ldots,\ \ell$; hence $c \geq \ell$, but $\ell \geq k \geq c$; therefore, in this case $c = \ell = k$, B is the empty set and $m = 0$.

Consider the outdirected tree $\overrightarrow{T}$ with root r_1. Clearly R and B are disjoint and $(R \cup B) \cap V_1(T) = \emptyset$. If $r_i r_j$ is an arc of $\overrightarrow{T}$ then $T' = (T - r_i r_j) \cup \pi_i \cup \pi_j$ is a $(s+1,\ c)$-subtree of G containing more vertices than T. If $b_i r_j$ is an arc of $\overrightarrow{T}$ then $T' = (T - b_i r_j) \cup \pi_1 \cup \pi_j$ is a $(s+1, c)$-subtree of G larger than T. Therefore we also have $N^+(R \cup B) \cap R = \emptyset$.

Choose $X(B)$ and ϕ as in lemma 1 and let W be the range of ϕ. Since ϕ is one to one then:

$$\begin{aligned}
|W| &= |N^+(R) \cup (N^+(B) \setminus X(B))| \\
&= |N^+(R)| + |N^+(B) \setminus X(B)| \\
&= [1 + t(s-1) + (\ell - t)s] + [(c - \ell + t)(s-1)] \\
&= 1 + \ell(s-1) + (\ell - t) + (c - \ell + t)(s-1) \\
&\geq 1 + \ell(s-1) + c \\
&\geq 1 + k(s-1) + c.
\end{aligned}$$

Since $\alpha \leq 1 + k(s-1) + c$ then $W \cup \{w\}$ cannot be an independent set in G, hence there is an edge $xy \in E(G)$ with both endvertices in $W \cup \{w\}$. In addition, by the choice of P any vertex in T adjacent in G to w must lie in R but $R \cap W \subset R \cap (V_1(T) \cup N^-(R)) = \emptyset$ so both x and y must be in W. By lemma 1, W is independent in T, then $xy \in E(G) \setminus E(T)$.

Let u and v in $N^+(R) \cup (N^+(B) \setminus X(B))$ be such that $\phi(u) = x$ and $\phi(v) = y$. By condition 3 in lemma 1 we can choose a vertex $p(x,y) \in \{u^-, v^-\} \cap V(T_{xy})$.

We prove that T may be transformed into a tree T' containing all vertices of T and the vertex w. The tree T' is formed by deleting some edges of T, breaking it into several components. We then join these components by adding the edge xy and some of the w to T paths in P.

This procedure is done in such a way that when an edge is added to a vertex λ of T, then either λ is a terminal vertex of T; another edge incident with λ is deleted or λ is r_1. In the latter case then $d_T(r_1) = s$, $d_T(p(x,y)) = s+1$ and an edge of T incident with $p(x,y)$ has been deleted. Hence the maximum degree and the number of vertices with degree $s+1$ is unchanged from T to T'. Several cases must be considered.

Let z be a vertex in $V(T_{xy})$ adjacent to $p(x,y)$ and for each $\alpha \in N^-(R)$ let $i(\alpha)$ be such that $\alpha r_{i(\alpha)}$ is an arc of $\overrightarrow{T}$.

Case 1. $x \in V_1(T)$ and $y \in V_1(T)$.

If $p(x,y) \in R$, say $p(x,y) = r_i$, then
$T' = ((T + xy) - r_i z) \cup \pi_i$

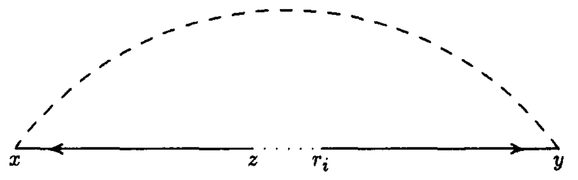

Fig. 1.

Otherwise $p(x,y) = b_i$ for some i and then
$T' = ((T + xy) - b_i z) \cup \pi_1$

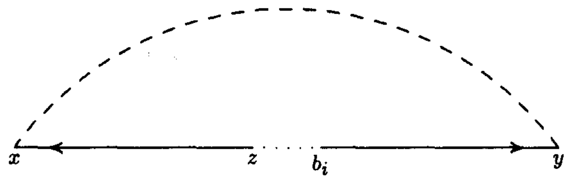

Fig. 2.

Case 2. $x \in V_1(T)$ and $y \in N^-(R)$.
If $r_{i(y)} \in V(T_{xy})$ then
$T' = \left((T + xy) - yr_{i(y)}\right) \cup \pi_{i(y)}$

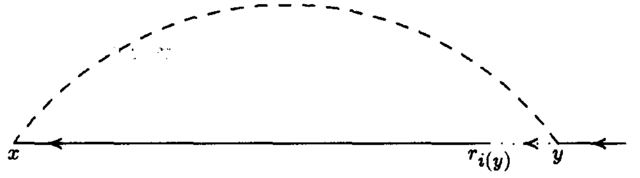

Fig. 3.

Otherwise $r_{i(y)} \notin V(T_{xy})$, in which case when $p(x,y) \in R$, say $p(x,y) = r_j$ then

$T' = \left(((T + xy) - r_j z) - yr_{i(y)}\right) \cup \pi_j \cup \pi_{i(y)}$

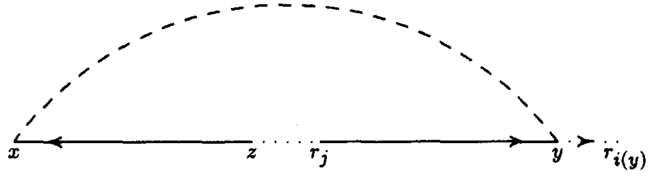

Fig. 4.

and when $p(x,y) = b_j$ for some j then

$T' = \left(((T + xy) - b_j z) - yr_{i(y)}\right) \cup \pi_1 \cup \pi_{i(y)}$

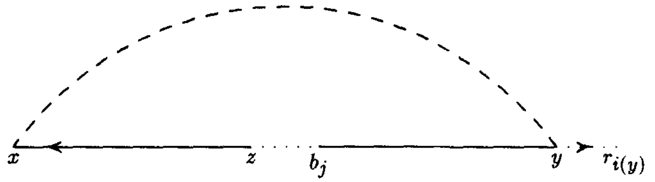

Fig. 5.

Case 3. $x \in N^-(R)$ and $y \in V_1(T)$.

In this case T' is constructed by interchanging x and y in case 2.

Case 4. $x \in N^-(R)$ and $y \in N^-(R)$.

If either $r_{i(x)}$ or $r_{i(y)}$ lies in $V(T_{xy})$ then

$$T' = \Big(((T + xy) - xr_{i(x)}) - yr_{i(Y)}\Big) \cup \pi_{i(x)} \cup \pi_{i(y)}.$$

Fig. 6.

Otherwise $r_{i(x)} \notin V(T_{xy})$, $r_{i(y)} \notin V(T_{xy})$ in which case when $p(x,y) \in R$, say $p(x,y) = r_h$ then

$$T' = \Big(\Big(((T + xy) - r_h z) - xr_{i(x)}\Big) - yr_{i(y)}\Big) \cup \pi_h \cup \pi_{i(x)} \cup \pi_{i(y)}.$$

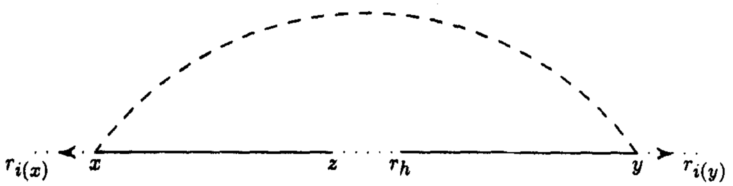

Fig. 7.

and when $p(x,y) = b_h \in B$ then

$$T' = \Big(\Big(((T + xy) - b_h z) - xr_{i(x)}\Big) - yr_{i(y)}\Big) \cup \pi_1 \cup \pi_{i(x)} \cup \pi_{i(y)}.$$

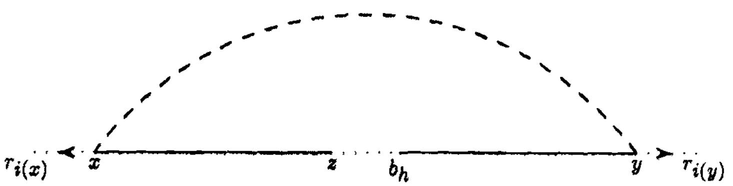

Fig. 8.

Cases 1 to 4 cover all possibilities and in each case T' is a $(s+1,\ c)$-subtree of G larger than T which contradicts the choice of T; therefore T is a spanning $(s+1,\ c)$-subtree of G. ∎

Theorem 2 is best possible in the sense that for each k, s and c with $0 < k$, $3 \leq s$ and $0 \leq c \leq k$, the complete bipartite graph $F = K_{k,2+k(s-1)+c}$ is k-connected, has independence number $\alpha = 2 + k(s-1) + c$ and:
(a) If $c < k$ then all spanning trees of F with degrees not exceeding $s+1$ contain at least $c+1$ vertices of degree $s+1$.
(b) If $c = k$ then all spanning trees of F have at least one vertex of degree larger than $s+1$. ∎

For the sake of completness we include the following weaker but more comprehensive result.

Theorem 3. *Let G be a k-connected graph with independence number α. If $\alpha \leq 1+ks$ for some positive integer s then G has a spanning tree with maximum degree at most $s+1$.*

Proof. Due to theorem 2 and the result by Chvátal and Erdős we only need to prove the case $s = 2$.

Let T be a subtree of G with maximum degree less than 4 and having as many vertices as possible. Again if T is not a spanning tree let w be a vertex of G not in T and $P = \{\pi_1,\ \pi_2,\ \ldots,\ \pi_\ell\}$ be a maximum collection of paths, pairwise disjoint apart from w, starting at w and terminating in $R = \{r_1,\ r_2,\ \ldots,\ r_\ell\}$ with $V(\pi_i) \cap V(T) = \{r_i\}$. By Menger's theorem $\ell \geq k$ and by the choice of T, $d_T(r_i) = 3$ for every $i = 1,\ \ldots,\ \ell$.

We now apply lemma 1, with $B = \emptyset$ and $\overrightarrow{T}$ outdirected with root r_1. As before we can find an edge xy of G not in T with both endvertices in $W = \phi[N^+(R)]$. Cases analogous to 1, 2, 3 and 4 are considered and a tree T' may be constructed such that $d_{T'}(u) \leq 3$ for every $u \in V(T')$ and $V(T) \cup \{w\} \subset V(T')$. The graphs $K_{k,2+sk}$ with s and k positive show that Theorem 3 is also best possible. ∎

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